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PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR GRADUATION WITH HONORS RESEARCH DISTINCTION IN MATHEMATICS IN THE UNDERGRADUATE COLLEGES OF THE OHIO STATE UNIVERSITY

BY PAUL APISA

THE OHIO STATE UNIVERSITY
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PROJECT ADVISOR: PROFESSOR RONALD SOLOMON, DEPARTMENT OF MATHEMATICS

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1. Introduction

Let G be a group. A generating sequence $(v_1, ..., v_n)$ of G is said to be minimal if any proper subsequence fails to generate G. In the case of vector spaces, the minimal generating sequences are exactly the (ordered) bases. Two properties that vector spaces possess with respect to group generation (which not all groups possess) are

(1) Every minimal generating sequence (basis) is the same size
(2) $\mathrm{Aut}(V)$ acts transitively on the set of minimal generating sequences.

Groups with property (1) are called B-groups and groups with property (2) are called UMP groups. It is clear that UMP groups are B groups. By [1]

Theorem 1.1. *G is a B group if and only if G is a p-group or $G \cong P \rtimes Q$ where P is a p-group and Q is a cyclic q-group, for distinct primes p and q, such that $Q/C_Q(P)$ acts fixed point freely on P.*

Since $\mathrm{Aut}(G)$ acts transitively on the set of generators, all the generators of a UMP group have the same order. Since a B-group of the form $P \rtimes Q$ has generators of order p^n and q^m, these groups cannot be UMP. Therefore, all UMP groups are p-groups.

In section 2 of this paper, a general means of constructing UMP groups is presented. As a corollary, the Burnside groups on n letters of exponent p^m are shown to be UMP. In section 3, we study quotients and maximal subgroups of UMP groups. In section 4, the central automorphisms of UMP groups are classified, which leads to a complete classification of class-2 UMP groups with cyclic center in section 5. These groups are the Heisenberg groups modulo p^n when p is odd and hyperquaternion groups when $p = 2$. In section 6, the commutator subgroup of class-2 UMP groups is identified. In the final section, I have collected a list of UMP groups from GAP.

2. Homogeneous Groups and Ultracharacteristic Subgroups

The results which I have collected in this section, I learned from Dan Collins and Keith Dennis, who in turn drew on the work of Gaschutz and the Neumanns. [2] is a good reference for these results.

Let G be a group. Let $\Gamma_n(G)$ denote the set of ordered generating sequences of length n. Let F_n denote the free group on n letters. For each $s = (g_1, ..., g_n) \in \Gamma_n(G)$, let $\pi_s : F_n \longrightarrow G$ such that $\pi_s : x_i \longrightarrow g_i$. If $H \trianglelefteq G$, then H is said to be ultracharacteristic if for all $N \trianglelefteq G$ such that $G/N \cong G/U$ we have that $N \leq U$.

Lemma 2.1. *Let $U \trianglelefteq G$ be an ultracharacteristic subgroup and $N \trianglelefteq G$ such that $G/N \cong G/U$, then $N = U$.*

Proof. Suppose to a contradiction that $N < U$ and let $h : G/N \longrightarrow G/U$ be an isomorphism. Therefore, $h(U/N)$ is a nontrivial subgroup of G/U, so there is a subgroup $U < K \leq G$ such that $K/U = h(U/N)$. However,

$$G/K = \frac{G/U}{K/U} = \frac{h(G/N)}{h(U/N)} \cong \frac{G/N}{U/N} = G/U$$

which contradicts the fact that U is ultracharacteristic. $\square$

Now we say that a group G is "homogeneous in rank n" for $n \geq r(G)$ if $\mathrm{Aut}(G)$ acts transitively on $\Gamma_n(G)$ and that $H(n, G) = F_n / \cap_{s\in\Gamma_n(G)} K_s$ is the "homogeneous cover of rank n".

Lemma 2.2. *$K = \cap_{s\in\Gamma_n(G)} K_s$ is ultracharacteristic.*

Proof. Let $L \subseteq F_n$ such that $h : F_n/L \longrightarrow F_n/K$ is an isomorphism.

$$\begin{tikzcd} F_n \arrow[r, two heads, "\pi_K"] \arrow[d, two heads, "\pi_L"'] & F_n/K \arrow[r, two heads, "\bar{\pi}_s"] & G \\ F_n/L \arrow[ur, two heads, "h"] & & \end{tikzcd}$$

Let σ be a map from $\Gamma_n(G)$ to itself such that $\sigma(s) = t$ iff $\pi_t = \bar{\pi}_s h \pi_L$. Then $\sigma(s) = \sigma(t)$ iff $\bar{\pi}_t h \pi_L = \bar{\pi}_s h \pi_L$ iff $t = s$ since $h\pi_L$ is a surjection. Since $\Gamma_n(G)$ is a finite set, σ is a bijection. Since $L = \ker \pi_L \subseteq \ker\left(\bar{\pi}_s h \pi_L\right) = K_{\sigma(s)}$, we have that $L \subseteq \cap_{s\in\Gamma_n(G)} K_{\sigma(s)} = K$ as desired. $\square$

Lemma 2.3. *$K_s = K_t$ iff s and t are in the same orbit of* $\mathrm{Aut}(G)$.

Proof. If $K_s = K_t$, then $h : F_n/K_s \cong F_n/K_t$ and so $h \in \mathrm{Aut}(G)$ takes s to t. If $h \in \mathrm{Aut}(G)$ takes s to t, then $h\pi_s = \pi_t$ and thus $K_s = \ker \pi_s = \ker \pi_t = K_t$ since h is an isomorphism. $\square$

Corollary 2.4. *The following are equivalent:*

1. *For any (and hence all) $s \in \Gamma_n(G)$, $K_s = \ker \pi_s$ is ultracharacteristic.*
2. *$K_s = K_t$ for all $s, t \in \Gamma_n(G)$*
3. *$H(n, G) = G$*
4. *G is homogeneous in rank n.*

Notice that if $G = F_n/K$ where K is an ultracharacteristic subgroup, then K is the unique set of relations that define G, i.e. if $G = F_n/U$, then $U = K$. The previous lemma implies that groups with such a unique presentation have the additional property that $\mathrm{Aut}(G)$ acts transitively on the set of $r(G)$ length generating sequences.

Corollary 2.5. *$H(n, G)$ is homogeneous of rank n and $r(H) = n$.*

Proof. Suppose to a contradiction that $r(H) < n$ and let $s \in \Gamma_{n-1}(H)$. Then (s, h) is in the same orbit as $(s, 1)$, which is a contradiction. It is homogeneous since $K_{\bar{s}} = K$ for all $s \in \Gamma_n(G)$. $\square$

Proposition 2.6. *For every epimorphism $f : H \longrightarrow G$ where H is homogeneous of rank n there is a lift $\bar{f} : H \longrightarrow H(n, G)$ such that $\pi \circ \bar{f} = f$.*

Proof. Let $t \in \Gamma_n(G)$ and let $\hat{t}$ be its lift to H. Then

$$\begin{tikzcd} F_n \arrow[r, two heads, "\bar{\pi}_t"] \arrow[d, two heads, "\pi_{\hat{t}}"'] \arrow[dr, two heads, "\pi_t"] & H(n,G) \arrow[d, two heads, "\pi"] \\ H \arrow[r, two heads, "f"] & G \end{tikzcd}$$

Since $\ker \pi_{\hat{t}} \leq \ker \pi_t$, $\bar{\pi}_t$ factors through $\pi_{\hat{t}}$ as desired.

$\square$

2.1. **Subdirect Products.** A group G is said to be the subdirect product of the groups $(G_i)_{i \in I}$ if there is an injection $f : G \longrightarrow \prod_{i \in I} G_i$

$$\begin{tikzcd} G \arrow[r, hook, "f"] \arrow[dr, two heads] & \prod\limits_{i \in I} G_i \arrow[d, two heads, "\pi_i"] \\ & G_i \end{tikzcd}$$

such that $\pi_i \circ f$ is a surjection for each $i \in I$.

Lemma 2.7. *Let H be a subdirect product of k copies of G. Then*

(1) $|H| \leq |G|^k$

(2) *$exp(H) = exp(G)$*

(3) *H is abelian iff G is*

(4) *H is nilpotent iff G is (in which case both have the same nilpotency class).*

Proposition 2.8. *$H(n, G)$ is the subdirect product of $h_n(G)$ copies of G, where $h_n(G)$ is the number of orbits that* $\mathrm{Aut}(G)$ *has on $\Gamma_n(G)$.*

Proof. Let $(s_i)_{i=1}^k$ be a collection of representatives of the orbits in $\Gamma_n(G)$ under $\mathrm{Aut}(G)$ action where $k = h_n(G)$. Let $\bar{f} : F_n \longrightarrow G^k$ be the map

$$\bar{f}(x) = \left(\pi_{s_1}(x), ..., \pi_{s_k}(x)\right)$$

Then $\ker \bar{f} = \cap_{i=1}^k K_{s_i} = K$ where $F_n/K = H(n, G)$; therefore, $\bar{f}$ descends to an injection $f : H(n, G) \longrightarrow G^k$. Since projecting onto each coordinate maps into the image of π_{s_i}, we are done. $\square$

3. Groups with the Universal Mapping Property

A group G is said to possess the universal mapping property, i.e. be a UMP group, if $\mathrm{Aut}(G)$ acts transitively on the set of irredundant generating sequences. Since UMP groups are B groups, by the classification of B groups, UMP groups are p-groups. This observation shows that a group G is UMP iff it is a homogeneous p-group. Notice that UMP groups possess a high degree of symmetry in that all generators have the same order.

In this section, two examples of UMP groups are given, quotients of UMP groups are discussed, and basic results on class-2 UMP groups are shown.

3.1. Examples of UMP Groups.

Proposition 3.1. *Finite Burnside groups of exponent p^n, $B(m,p^n)$ are UMP.*

Proof. Let $K \trianglelefteq F_m$ be the normal subgroup generated by words of the form w^{p^n}. Then $B(m,p^n) = F_n/K$. Moreover, for any generating sequence $s \in \Gamma_n(G)$, the map π_s must contain K in its kernel. Since F_n/K is finite, $K = \pi_s$. Thus, K is ultracharacteristic and $B(m,p^n)$ is UMP. $\square$

Proposition 3.2. *A UMP group is abelian iff it is a homocyclic abelian p-group.*

Proof. For abelian p-groups G, $\Phi(G) = G^p$. Thus, the generators are precisely the elements of highest order. Hence, $G = \mathbb{Z}_{exp(G)}^{r(G)}$. $\square$

3.2. Quotients of UMP Groups.

Lemma 3.3. *If N is a proper characteristic subgroup of a UMP group G, then $N \leq \Phi(G)$ and G/N is UMP.*

Proof. Suppose that $N \leq G$ contains a generator x. If $y \in G \backslash N$ is a generator, then there is an automorphism that sends x to y. Since N is characteristic, $y \in N$. Therefore, $N = G$. Hence, if N is a proper characteristic subgroup of G, $N \leq \Phi(G)$. In this case, $\Gamma_n(G/N) = \Gamma_n(G)/N$. Moreover, each element of $\mathrm{Aut}(G)$ descends to an automorphism of $\mathrm{Aut}(G/N)$ and hence G/N is UMP. $\square$

Similarly, we see that if $H = \langle x_1, ..., x_n \rangle$ is a sequence of generators of G, then G/H is still UMP.

Corollary 3.4. *G/G' is homocyclic abelian*

If G is a p-group of rank $n = \log_p(G) - \log_p(|\Phi(G)|)$, then G is UMP iff $|Aut(G)| = \Phi(G)^n \prod_{k=0}^{n-1}(p^n - p^k)$ since that is the number of lifts of bases of $G/\Phi(G) = \mathbb{F}_p^n$ (or similarly since every element of $GL_n(\mathbb{F}_p)$ has exactly $|\Phi(G)|^n$ lifts into $\mathrm{Aut}(G)$).

Proposition 3.5. *All the maximal subgroups of a UMP group are automorphic*

Proof. Any maximal subgroup projects onto a unique $n-1$-subspace of $G/\Phi(G)$, which fully determines it (since the maximal subgroups all contain the Frattini subgroup). These subspaces are permutable by automorphisms in G, since G is UMP. Thus, the maximal subgroups are automorphic. □

Proposition 3.6. *For nonabelian UMP groups, no generators commute. In particular,* $Z(G) < \Phi(G)$*.*

3.3. **Elementary Results for Class-2 UMP Groups.**

Proposition 3.7. *If G is class-2, then $[\cdot, x] : G \to [G, G]$ is a group homomorphism for all $x \in G$. Furthermore,* $\ker[\cdot, x] = C_x(G)$*.*

Proof.

$$[gh, x] = ghxh^{-1}g^{-1}x^{-1} = g(hxh^{-1}x^{-1})xg^{-1}x^{-1} = [g, x] \cdot [h, x]$$

□

Corollary 3.8. *If G is class-2 UMP, then $[G, G]$ has exponent p iff $Z(G) = \Phi(G)$.*

Proof. Since $[x, y]^p = [x, y^p]$, $[G, G]$ has exponent p iff $[x, y^p] = 1$ for all $x, y \in G$ iff $G^p \leq Z(G)$ iff $Z(G) = \Phi(G)$. □

Theorem 3.9 (Solomon)**.** *The only extraspecial p-groups that are UMP are Q_8 and $B(3, p)$ - the Burnside group of exponent p generated by three elements.*

Proof. All other extraspecial p-groups have generators that commute. □

4. Central Automorphisms of UMP Groups

Let G be a group and let A denote its automorphism group. The central automorphisms of G are defined to be $A_c = C_A(Inn(G))$. To fix some notation, for $g \in G$, let $\phi_g \in A$ be the automorphism that acts by conjugating by g, i.e. $\phi_g(x) = gxg^{-1}$.

Before proceeding, let's make a quick observation. Suppose that G is an n-generated UMP group. Then $Z(G)^n$ embeds in A. To see this, pick a minimal generating sequence for G, $(v_1, ..., v_n)$. To each $(z_1, ..., z_n) \in Z(G)^n$ associate the automorphism that sends

$$(v_1, ..., v_n) \longrightarrow (v_1 z_1, ..., v_n z_n)$$

This map is an embedding of $Z(G)^n$ in A. Notice too that $Z(G)^n$ fixes each element of G'.

Lemma 4.1. $A_c(G) = Z(G)^n$ *for an n-generated UMP group G.*

Proof. Given an ordered generating set $(v_1, ..., v_n)$, the inner automorphism ϕ_g is given by

$$(v_1, ..., v_n) \xrightarrow{\phi_g} ([g, v_1]v_1, ..., [g, v_n]v_n)$$

Now suppose that $\phi \in A_c$, then for all $g \in G$,

$$\begin{array}{ccc} v & \xrightarrow{\phi_g} & [g,v]v \\ \Big\downarrow \phi & & \Big\downarrow \phi \\ \phi v & \xrightarrow{\phi_g} & [\phi g, \phi v]\phi v = [g, \phi v]\phi v \end{array}$$

Therefore, for each generator v and for each $g \in G$ we have

$$[\phi g, \phi v] = [g, \phi v] \text{ i.e. } \phi_{\phi(g)} = \phi_g$$

therefore ϕ induces the identity on $G/Z(G)$ and so $\phi \in Z(G)^n$. This shows that $A_c \subseteq Z(G)^n$. For the reverse inclusion, let $z = (z_1, ..., z_n) \in Z(G)^n$. Since z fixes each element of the commutator subgroup we have

$$\begin{array}{ccc} v_i & \xrightarrow{\phi_g} & [g,v_i]v_i \\ \Big\downarrow z & & \Big\downarrow z \\ v_i z_i & \xrightarrow{\phi_g} & [g, v_i]v_i z_i \end{array}$$

and so z commutes with each ϕ_g as desired.

□

Notice that as a consequence of this proof, $Inn(G) \leq (G')^n$. The next result will rely on a theorem of M.J. Curran and D.J. McCaughan in [3],

Theorem 4.2 ([3])**.** *If G is a class-2 p-group with cyclic center, then $[Z(G) : G'] = [A_c : Inn(G)]$.*

Corollary 4.3. *If G is a class-2 (n-generated) UMP group with cyclic center, then $|G| = |Z(G)|^{n+1}$ and $Z(G) = G'$.*

Proof. Since $Inn(G) \leq (G')^n \leq Z(G)^n = A_c$, by theorem 1.2,

$$[Z(G) : G'] = [A_c : Inn(G)] \geq [Z(G) : G']^n$$

Since $n > 1$, $[Z(G) : G'] = 1$. Another application of theorem 1.2 yields,

$$\frac{|Z(G)|^n}{|G|/|Z(G)|} = [A_c : Inn] = [Z(G) : G'] = 1$$

which implies that $|G| = |Z(G)|^{n+1}$. □

5. Classification of Class-2 UMP Groups with Cyclic Centers

The preceding result implies that the structure of a class-2 UMP group G with $Z(G) = Z_{p^m}$ is

$$G = Z_{p^m} : Z_{p^{m-1}}^n : \mathbb{F}_p^n$$

where the first factor is the center (and also commutator subgroup), the second factor is the remainder of the Frattini subgroup (generated by $\Omega_1(G)$), and the final factor is the quotient of G by the Frattini subgroup.

Lemma 5.1. *Let G be a class-2 UMP group with cyclic derived subgroup, then $G' = \langle [v_1, v_2] \rangle$.*

Proof. Since $[\cdot, x] : G \longrightarrow G'$ is a homomorphism (and similarly for $[x, \cdot]$) the commutators $[v_i, v_j]$ for $i \neq j$ generate G'. However, by the UMP property, all of these have the same order in G' and thus are all generators. In particular, any element $[v_i, v_j]$ for $i \neq j$ generates G' since G' is cyclic. □

Lemma 5.2. *Let G be a class-2 UMP group with cyclic derived subgroup iff G is 2-generated.*

Proof. Suppose to a contradiction that G' is cyclic and G is at least 3 generated. Since $\langle [v_1, v_3] \rangle = G'$, there is an integer k such that $[v_1, v_3^k] = [v_1, v_3]^k = [v_1, v_2]^{-1}$. Hence, $[v_1, v_2 v_3^k] = 1$ contrary to the fact that no two generators commute. □

Lemma 5.3. *Let G be a class-2 UMP p-group with cyclic center. Let e be the exponent of the center of G. Then p is odd and G has exponent e or $p = 2$ and the exponent of G is $2e$.*

Proof. By corollary 1.3 and lemma 2.2,

$$G = \mathbb{Z}_{p^m} : \mathbb{Z}_{p^{m-1}}^2 : \mathbb{F}_p^2$$

where the first factor is $\mathbb{Z}_{p^m} = Z(G) = G'$. Let $c = [a, b]$. Then we have that

$$a^{p^m} = [a, b]^k \qquad b^{p^m} = [a, b]^\ell \qquad (ab)^{p^m} = a^{p^m} b^{p^m} c^{\frac{p^m(p^m-1)}{2}}$$

By the UMP property, there is an automorphism $\sigma : G \longrightarrow G$ such that $\sigma : a \longrightarrow b$ and $\sigma : b \longrightarrow a$, which shows that

$$c^\ell = b^{p^m} = \sigma(a)^{p^m} = \sigma([a, b])^k = [b, a]^k = c^{-k}$$

$$b^{p^m} a^{p^m} = 1$$

When p is an odd prime, c has order p^m and $2|p-1$. Thus, $(ab)^{p^m} = a^{p^m} b^{p^m} = 1$. Since ab is a generator, all generators must have order p^m and hence G has exponent p^m.

Now suppose that $p = 2$. By the same argument as just used

$$(ab)^{p^m} = c^{2^{m-1}(2^m-1)} = c^{-2^{m-1}} = c^{2^{m-1}}$$

which is the unique element of order 2 in the center. Therefore, $v^{p^m} = c^{2^{m-1}}$ for any generator v of G.

□

Proposition 5.4. *If G is a class-2 UMP group with cyclic center, then there is an integer $m > 1$ such that G is the Heisenberg group over $\mathbb{Z}_{p^m}$, i.e. 3×3 upper triangular matrices with entries in $\mathbb{Z}_{p^m}$, when p is odd or*

$$Q_{8^m} = \langle a, b, c | a^{2^m} = b^{2^m} = c^{2^{m-1}}, |c| = 2^m, [a, b] = c, [a, c] = [b, c] = 1 \rangle$$

when $p = 2$.

Proof. First let p be an odd prime. By corollary 1.3, lemma 2.2, and lemma 2.3,

$$G = \mathbb{Z}_{p^m} : \mathbb{Z}_{p^{m-1}}^2 : \mathbb{F}_p^2$$

where the first factor is $\mathbb{Z}_{p^m} = Z(G) = G'$ and p^m is the exponent of the group. To give a more concrete group presentation

$$G = \langle a, b, c | a^{p^2} = b^{p^2} = c^{p^2} = [a, c] = [b, c] = e, [a, b] = c \rangle$$

Phrased a little differently, each word in G is expressible as $a^k b^\ell c^m$ with multiplication given by

$$(k, \ell, m) \cdot (k', \ell', m') = (k + k', \ell + \ell', m + m' + \ell k')$$

from which it is clear that G is isomorphic to the specified Heisenberg group under the identifications:

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When $p = 2$, the same three results used above give the group presentation.

□

Proposition 5.5. *The Heisenberg groups modulo p^n (where p is an odd prime) are class-2 UMP with cyclic center.*

Proof. Consider a map given by

$$\sigma : a \longrightarrow a^{n_1} b^{n_2} c^{n_3}$$

$$b \longrightarrow a^{m_1} b^{m_2} c^{m_3}$$

Now consider the map $A = \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$. Let $\widetilde{A}$ be A with coefficients mod p, i.e. the map that σ induces on $G/\Phi(G)$. As we are only interested in automorphisms of G, we only have to worry about σ such that $\det A \mod p = \det \widetilde{A} \not\equiv 0 \mod p$. Therefore,

$$\sigma(c) = [\sigma(b), \sigma(a)] = [b^{m_2}, a^{n_1}][a^{m_1}, b^{n_2}] = c^{\det A}$$

which is still a generator of $Z(G)$ since $\det A \not\equiv 0 \mod p$. Now for a general element of G, σ acts by

$$\begin{aligned}\sigma\big(a^{k_1}b^{k_2}c^{k_3}\big) &= \big(a^{n_1}b^{n_2}c^{n_3}\big)^{k_1}\big(a^{m_1}b^{m_2}c^{m_3}\big)^{k_2}c^{k_3\det A}\\ &= a^{k_1n_1+k_2m_1}b^{k_1n_2+k_2m_2}c^{f(k)}\end{aligned}$$

where

$$f(k) = k_1n_3 + k_2m_3 + k_3\det A + n_1n_2T_{k_1-1} + m_1m_2T_{k_2-1} + k_1k_2m_1n_2$$

such that T_n is the nth triangular number. Consider two elements of G, $k = (k_1, k_2, k_3)$ and $k' = (k_1', k_2', k_3')$ where

$$k \cdot k' = (k_1 + k_1', k_2 + k_2', k_3 + k_3' + k_1'k_2)$$

and where

$$\sigma(k) = (k_1n_1 + k_2m_1, k_1n_2 + k_2m_2, f(k))$$

$$\sigma(k \cdot k') = \big((k_1 + k_1')n_1 + (k_2 + k_2')m_1, (k_1 + k_1')n_2 + (k_2 + k_2')m_2, f(k \cdot k')\big)$$

A quick computation shows that

$$\begin{aligned}f(k \cdot k') &= f(k) + f(k') + n_1n_2k_1k_1' + m_1m_2k_2k_2' + k_1'k_2(n_1m_2 - m_1n_2) + (k_1k_2' + k_1'k_2)m_1n_2\\ &= f(k) + f(k') + (k_1'n_1 + k_2'm_1)(k_1n_2 + k_2m_2)\end{aligned}$$

and thus

$$\sigma(k \cdot k') = \sigma(k) \cdot \sigma(k')$$

Thus, σ is a homomorphism when $\widetilde{A}$ is invertible and indeed an automorphism.

□

A similar computation, which I have elected not to include, shows that Q_{8^m} are UMP too.

6. Commutator Subgroups of Class-2 UMP Groups

The exterior square of a vector space is defined to be $\Lambda^2(V) = V \wedge V = V \otimes V/W$ where $W \leq V$ is generated by words of the form $w \otimes w$. Similarly, for a commutative ring R, define $\Lambda^2(R) = R \otimes R/W$ where $W \leq V$ is generated by words of the form $w \otimes w$.

Theorem 6.1. *Let G be a class-2 UMP p-group for odd prime p, then $G' = \Lambda^2(\mathbb{Z}_{p^m}^n)$ where p^m is the exponent of G'.*

Proof. It suffices to show that there is no solution to

$$[v_n, w] + \sum_{1 \leq i < j \leq n-1} k_{ij}[v_i, v_j] = 0$$

where w is a generator and is a word that is formable without the use of v_n. By applying an automorphism to switch w and v_1, this is equivalent to showing that there is no solution to

$$[v_n, v_1] = \sum_{1\le i<j\le n-1} k_{ij}[v_i, v_j]$$

Applying an automorphism to switch v_n and v_1 yields

$$-\sum_{1\le i<j\le n-1} k_{ij}[v_i, v_j] = -[v_n, v_1] = [v_1, v_n] = \sum_{i=2}^{n-1} k_{1i}[v_n, v_i] + \sum_{2\le i<j\le n-1} k_{ij}[v_i, v_j]$$

Thus,

$$[v_1^{-1}v_n, v_n^{-2}] = [v_1, v_n]^2 = \sum_{i=2}^{n-1} k_i[v_n v_1^{-1}, v_i] = [v_n v_1^{-1}, v_2^{k_2} \cdot \ldots \cdot v_{n-1}^{k_{n-1}}]$$

from which we see that

$$0 = [v_n v_1^{-1}, v_2^{k_2} \cdot \ldots \cdot v_{n-1}^{k_{n-1}} v_n^2]$$

which contradicts the fact that no two generators commute. □

Recall that for a group G, we let $V = G/\Phi(G)$ and $W = [G,G]/\Phi([G,G])$. In the class-2 case, $W = \Lambda^2(V)$. The following proof was suggested by Richard Lyons.

Lemma 6.2. *$\Lambda^2(V)$ is an irreducible $GL(V)$-module when V is an $\mathbb{F}_p$ vector space for odd prime p.*

Proof. Let $D \le GL(V)$ be the subgroup of diagonal matrices. Then $\Lambda^2(V)$ is the direct sum of one-dimensional D-invariant submodules (generated by $v_i \wedge v_j$ for $i \ne j$). These subspaces are non-isomorphic since $p \ne 2$ (when $p = 2$ there are no nonidentity diagonal matrices in $GL(V)$). However, the action of the permutation matrices P show that there are no DP -invariant subspaces. □

Corollary 6.3. *Let G be a class-2 UMP p-group where p is odd with elementary abelian Frattini subgroup. Let $\Omega_1(G)$, i.e. the subgroup generated by elements of the form g^p. Then $\Phi(G) = \Omega_1(G) \oplus G'$.*

Proof. It suffices to show that Ω_1 and G' nontrivially intersect. Since $\Omega_1(G)$ is an irreducible $GL(V)$-module (as it is isomorphic to V), it only intersects $\Lambda^2(V)$ nontrivially if $\Lambda^2(V) = \Omega_1(G)$. By considering dimensions, this only happens if $n = 3$. However, we then have (without loss of generality)

$$[v_1, v_2] = v_1^{k_1 p} v_2^{k_2 p} v_3^{k_3 p}$$

Hence, since there is a automorphism switching v_1 and v_2 and fixing v_3, we have that $k_3 = 0$ (since p is odd) and $k_1 = -k_2$ (let $k := k_1$ now). Since $(v_1 v_2^{-1})^p = v_1^p v_2^{-p}$,

$$\begin{aligned}
v_1 v_2 v_1^{-1} v_2^{-1} &= (v_1 v_2^{-1})^{kp} \\
v_2 v_1^{-1} &= v_2^{-1} (v_1 v_2^{-1})^{kp-1} v_1 \\
1 &= v_1 v_2^{-1} (v_1^{-1})(v_1 v_2^{-1})^{kp} v_1 \\
1 &= (v_1 v_2^{-1})^{kp+1} \\
1 &= v_1 (v_1 v_2^{-1})^{kp} v_2^{-1} \\
v_2^{kp+1} &= v_1^{kp+1}
\end{aligned}$$

The last equation is impossible since then v_1 and v_2 project onto the same element in the quotient by $\Phi(G)$. $\square$

Corollary 6.4. *Let G be a class-2 UMP p-group where p is odd. If $\Phi(G)$ is elementary abelian, then one of the two following possibilities holds:*

1. *$\Phi(G) = Z(G) = [G, G] = \Lambda^2(V)$ iff G has exponent p*
2. *$\Phi(G) = Z(G) = \Lambda^2(V) \oplus V$ iff G has exponent p^2.*

Proof. Since $\Phi(G)$ is elementary abelian, $exp(G) \leq p^2$. If G has exponent p, then Ω_1 is trivial and we are done. Therefore, suppose that G has exponent p^2. Since the generators have order p^2, $\Omega_1 = V$. Since G' has exponent p, $\Phi(G) = Z(G) = \Omega_1 \oplus \Lambda^2(V)$ as desired. $\square$

To summarize,

Corollary 6.5. *Let G be class-2 UMP group whose commutator subgroup has exponent p^e (p is odd).*

1. *$G' = \Lambda^2(\mathbb{Z}_{p^e}^n)$ and hence is cyclic iff G is 2-generated.*
2. *G' has exponent p iff $Z(G) = \Phi(G)$.*
3. *$Z(G)$ is cyclic iff G is a Heisenberg group.*
4. *$Z(G) = G'$ iff $G = \Lambda^2(\mathbb{Z}_{p^e}^n) : \mathbb{Z}_{p^e}^n$*
5. *$\Phi(G) = Z(G) = G'$ iff $exp(G) = p$.*
6. *$\Phi(G)$ is elementary abelian iff $Z = G' = \Phi(G)$ and $G = \Lambda^2(V) : V$ (when G has exponent p) or $G = \Lambda^2(V) \oplus V : V$ (and G has exponent p^2).*

7. Empirical Structure Results

7.1. **Class-2 2-groups.**

As shown in the analysis of class-2 2-groups with cyclic center, our only example of these groups will be the hyperquaternion groups of order 8^m with presentation

$$Q_{8^m} = \langle a, b, c | a^{2^m} = b^{2^m} = c^{2^{m-1}}, |c| = 2^m, [a, b] = c, [a, c] = [b, c] = 1 \rangle$$

All of the GAP accessible examples of UMP 2-groups are 2-generated.

(1) $G(8,4) = Q_8$
(2) $G(8^2,19) = Q_{8^2}$
(3) $G(2^7,5)$ has exponent 8 and cyclic $G' < Z(G) = \Phi(G)$ which have sizes 2 and 2^5 respectively and has the structure

$$G = G' \times \mathbb{Z}_4^2 : \mathbb{Z}_2^2$$

Each of the following sections lists the nonabelian UMP groups that are accessible with GAP and collects some facts about their structure. The notation $G(p^k, m)$ means that a group has order p^k and has index m in GAP's small group library.

7.2. **3-groups.**

(1) $G(3^3,3)$ is 2-generated, exponent 3, and has $Z(G) = \Phi(G) = [G,G]$, all of size 3. This is $B(2,3)$. Note that $B(n,3)$ has order $n + \binom{n}{2} + \binom{n}{3}$
(2) $G(3^5,2)$ is 2-generated, exponent 9, and has $[G,G] < Z(G) = \Phi(G)$. The center has size 27 versus the commutator, which has size 3.
(3) $G(3^6,24)$ is 2-generated, exponent 9, and has $Z(G) = [G,G] < \Phi(G)$. The center has size 9 and the Frattini subgroup has size 81.
(4) $G(3^6,122)$ is 3-generated, exponent 3, and has $Z(G) = \Phi(G) = [G,G]$ all of size 27.

7.3. **5-groups.**

(1) $G(5^3,3)$ is 2-generated, exponent 5, and has $Z(G) = \Phi(G) = [G,G]$, all of size 5. This cannot be $B(2,5)$.
(2) $G(5^5,2)$ is 2-generated, exponent 25, and has $[G,G] < Z(G) = \Phi(G)$ having size 5 and 125 resp.
(3) $G(5^5,3)$ is 2-generated, exponent 5, and has $Z(G) < [G,G] = \Phi(G)$ having size 25 and 125 resp.

Based on our previous data, we have that in fact $G(5^5,3)/Z(G) = G(5^3,3)$.

7.4. **7-groups.**

(1) $G(7^3,3)$ is 2-generated, exponent 7, and has $Z(G) = \Phi(G) = [G,G]$, all of size 7. This cannot be $B(2,5)$.
(2) $G(7^5,2)$ is 2-generated, exponent 49, and has $[G,G] < Z(G) = \Phi(G)$ having size 7 and 7^3 resp.
(3) $G(7^5,3)$ is 2-generated, exponent 7, and has $Z(G) < [G,G] = \Phi(G)$ having size 7^2 and 7^3 resp.

Based on our previous data, we have that in fact $G(7^5,3)/Z(G) = G(7^3,3)$. These groups are remarkably parallel to one another for all odd primes p.

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Paul Apisa, Department of Mathematics, The Ohio State University, Columbus, OH 43210

E-mail address: paul.apisa@gmail.com